

# Algebraic Connections vs. Algebraic $\mathcal{D}$ -modules: inverse and direct images.

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## Introduction

In the dictionary between the language of (algebraic integrable) connections and that of (algebraic)  $\mathcal{D}$ -modules, the operations of direct and inverse images for a smooth morphism are very important. To compare the definitions of inverse images for connections and  $\mathcal{D}$ -modules is easy. But the comparison between direct images for connections (the classical construction of the Gauss-Manin connection for smooth morphisms) and for  $\mathcal{D}$ -modules, although known to specialists, has been explicitly proved only recently in a paper of Dimca, Maaref, Sabbah and Saito in 2000 (see [DMMS]), where the authors' main technical tool was M. Saito's equivalence between the derived category of  $\mathcal{D}$ -modules and a localized category of differential complexes.

The aim of this short paper is to give a simplified summary of the [DMSS] argument, and to propose an alternative proof of this comparison which is simpler, in the sense that it does not use Saito equivalence. Moreover, our alternative strategy of comparison works in a context which is a precursor to the Gauss-Manin connection (at the level of  $f^{-1}\mathcal{D}_Y$ -modules, for a morphism  $f : X \rightarrow Y$ ), and may be of some intrinsic interest.

In section 1 we recall some generalities on connections and  $\mathcal{D}$ -modules. In section 2 we compare the operations of “inverse image” for connections and  $\mathcal{D}$ -modules. Section 3 is devoted to the comparison of the Gauss-Manin connection (in the case of smooth morphisms) with the notion of direct image for  $\mathcal{D}$ -modules: we supply a simplified summary of the [DMSS] argument. Finally, in the last section we propose our alternative proof of this comparison which does not use Saito equivalence.

## §1. Generalities on connections and $\mathcal{D}$ -modules

Let  $X$  be a smooth  $K$ -variety, where  $K$  is a field of characteristic 0. Following the terminology of [EGAIV, §16] we denote by  $\Omega_X^1$  the  $\mathcal{O}_X$ -module of differentials, by  $\mathcal{P}_X^1$  the  $\mathcal{O}_X$ -algebra of principal parts of order one: its two structures as  $\mathcal{O}_X$ -algebra will be referred to as the “left” and “right” structures, and tensor products will be specified by the position of the  $\mathcal{P}_X^1$  factor. We also use  $\mathcal{D}er_X$  or  $\Theta_X$  to denote the  $\mathcal{O}_X$ -module of derivations ( $\mathcal{O}_X$ -dual of  $\Omega_X^1$ , endowed with the usual structure of Lie-algebra), and  $\mathcal{D}_X$  to indicate the graded (left)  $\mathcal{O}_X$ -algebra of differential operators. On  $\mathcal{D}_X$  we consider the increasing filtration  $F$  defined by the order of differential operators. Then the associated graded  $\mathcal{O}_X$ -algebra, denoted by  $\text{Gr}\mathcal{D}_X$ , is commutative and it is generated (as  $\mathcal{O}_X$ -algebra) by  $\mathcal{D}er_X \subseteq F^1\mathcal{D}_X$ .

For any  $\mathcal{O}_X$ -module  $\mathcal{E}$  we will use the standard notation  $\mathcal{P}_X^1(\mathcal{E})$  for  $\mathcal{P}_X^1 \otimes_{\mathcal{O}_X} \mathcal{E}$ , where the tensor product involves the right  $\mathcal{O}_X$ -module structure of  $\mathcal{P}_X^1$ , while the  $\mathcal{O}_X$ -module structure is given by the left  $\mathcal{O}_X$ -module structure on  $\mathcal{P}_X^1$ .

### 1.1 Connections and $\mathcal{D}$ -modules

Let  $\mathcal{E}$  be an  $\mathcal{O}_X$ -module. The following supplementary structures on  $\mathcal{E}$  are equivalent:

- (i) an integrable connection, that is a morphism of abelian sheaves  $\nabla : \mathcal{E} \rightarrow \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{E}$  Leibniz w.r.t. sections of  $\mathcal{O}_X$  and such that  $\nabla^2 = 0$  for the natural extension of  $\nabla$  to the De Rham sequence;
- (ii) an  $\mathcal{O}_X$ -linear section  $\delta : \mathcal{E} \rightarrow \mathcal{P}_X^1 \otimes_{\mathcal{O}_X} \mathcal{E}$  of the canonical morphism  $\pi : \mathcal{P}_X^1 \otimes_{\mathcal{O}_X} \mathcal{E} \rightarrow \mathcal{E}$ ;
- (iii) an  $\mathcal{O}_X$ -linear Lie-algebra homomorphism  $\Delta : \mathcal{D}er_X \rightarrow \mathcal{D}iff_X(\mathcal{E})$  (for the usual Lie-algebra structures), where  $\mathcal{D}iff_X(\mathcal{E})$  is the sheaf of differential operators of  $\mathcal{E}$ ;
- (iv) a structure of left algebraic  $\mathcal{D}_X$ -module on  $\mathcal{E}$ .

The dictionary between these equivalent structures is well explained in [BO, 2.9, 2.11, 2.15]; let us give a sketch.

If  $c = c_X(\mathcal{E}) : \mathcal{E} \rightarrow \mathcal{P}_X^1 \otimes_{\mathcal{O}_X} \mathcal{E}$  denotes the canonical inclusion, then  $\delta = c + \nabla$  and  $\nabla = \delta - c$ .

For any  $\partial$  section of  $\mathcal{D}er_X$  the morphism  $\Delta$  is defined by  $\Delta_\partial = (\partial \otimes \text{id}) \circ \nabla$ , i.e.  $\Delta_\partial(e) = \langle \partial, \nabla(e) \rangle$ . On the other hand, the reconstruction of  $\nabla$  from  $\Delta$  requires a description using local coordinates  $x_i$  on  $X$  ( $dx_i$  and  $\partial_i$  are the dual bases of differentials and derivations): if  $e$  is a section of  $\mathcal{E}$ , then  $\nabla(e) = \sum_i dx_i \otimes \Delta_{\partial_i}(e)$ .

The morphism  $\Delta$  is equivalent to the data of a left  $\mathcal{D}_X$ -module structure on  $\mathcal{E}$  since it extends to a left action of  $\mathcal{D}_X$  on  $\mathcal{E}$  ([Bo, VI, 1.6]).

**1.2** A morphism of connections on  $X$  is an  $\mathcal{O}_X$ -linear morphism  $h : \mathcal{E} \rightarrow \mathcal{E}'$  compatible with the data, that is, such that  $\nabla' \circ h = (\text{id} \otimes h) \circ \nabla$ , or  $\delta' \circ h = (\text{id} \otimes h) \circ \delta$ , or equivalently  $\Delta'_\partial \circ h = h \circ \Delta_\partial$  for any  $\partial$  section of  $\mathcal{D}er_X$ , or finally which is  $\mathcal{D}_X$ -linear.

## §2. Inverse image for connections and $\mathcal{D}$ -modules

Let  $f : X \rightarrow Y$  be a finite type morphism of smooth  $K$ -varieties. For any  $\mathcal{O}_Y$ -module  $\mathcal{E}$ , let  $f^*(\mathcal{E}) = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{E}$  the inverse image by  $f$ .

### 2.1 Inverse image for connections

The easiest definition for the inverse image by  $f$  of a connection  $\mathcal{E}$  on  $Y$  is given in terms of  $\mathcal{O}_Y$ -linear maps. If  $\delta : \mathcal{E} \rightarrow \mathcal{P}_Y^1 \otimes_{\mathcal{O}_Y} \mathcal{E}$  defines the connection, let  $f^*\delta$  be the composition of the inverse image of  $\delta$  with the canonical morphism  $f^*(\mathcal{P}_Y^1) \rightarrow \mathcal{P}_X^1$ . Then we have a morphism

$$f^*\delta : f^*\mathcal{E} \rightarrow \mathcal{P}_X^1 \otimes_{\mathcal{O}_X} f^*\mathcal{E} \cong \mathcal{P}_X^1 \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{E} ,$$

which is clearly an  $\mathcal{O}_X$ -linear section of the canonical map  $\pi : \mathcal{P}_X^1 \otimes_{\mathcal{O}_X} f^*\mathcal{E} \rightarrow f^*\mathcal{E}$ .

An explicit description of the connection  $f^*\nabla$  on  $f^*\mathcal{E}$  can be given in the following way:

$$\begin{aligned} (f^*\nabla)(\alpha \otimes e) &= (f^*\delta - c_X)(\alpha \otimes e) \\ &= \alpha f^*\delta(1 \otimes e) - c_X(\alpha \otimes e) \\ &= \alpha f^*((\nabla + c_Y)(e)) - \mathbb{I} \otimes (\alpha \otimes e) \\ &= \alpha f^*(\nabla(e)) + \alpha(\mathbb{I} \otimes e) - \mathbb{I} \otimes (\alpha \otimes e) \\ &= \alpha \nabla(e) + \alpha \otimes 1 \otimes e - 1 \otimes \alpha \otimes e \\ &= \alpha \nabla(e) + d(\alpha) \otimes e \end{aligned}$$

(as usual,  $\alpha$  is a section of  $\mathcal{O}_X$  and  $e$  is a section of  $\mathcal{E}$ , or  $f^{-1}\mathcal{E}$ ).

### 2.2 Inverse image for $\mathcal{D}$ -modules

Let  $\mathcal{M}$  be a left  $\mathcal{D}_Y$ -module. The inverse image as  $\mathcal{O}$ -modules  $f^*\mathcal{M} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{M}$  locally admits an action of  $\mathcal{D}_X$  defined by

$$\alpha'(\alpha \otimes m) = (\alpha'\alpha) \otimes m \quad \text{and} \quad \partial(\alpha \otimes m) = \partial(\alpha) \otimes m + \alpha \left( \sum_i \partial(y_i) \otimes \eta_i(m) \right)$$

where  $\partial$  is a section of  $\mathcal{D}_X$ ,  $m$  a section of  $\mathcal{M}$  (or  $f^{-1}\mathcal{M}$ ),  $\alpha, \alpha'$  sections of  $\mathcal{O}_X$  ( $y_i$  local coordinates on  $Y$  and  $\eta_i$  the dual basis of  $dy_i$ ). These local definitions globalize to a  $\mathcal{D}_X$ -module structure on  $f^*\mathcal{M}$  [Bo, VI.4].

In this way the  $\mathcal{O}_X$ -module  $f^*\mathcal{D}_Y = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y$  is endowed with a structure of left  $\mathcal{D}_X$ -module, compatible with the obvious structure of  $f^{-1}\mathcal{D}_Y$ -module (by right multiplication). With this structure,  $f^*\mathcal{D}_Y$  is usually denoted by  $\mathcal{D}_{X \rightarrow Y}$  and the inverse image of a left  $\mathcal{D}_Y$ -module  $\mathcal{M}$  can be defined as

$$f^*\mathcal{M} = \mathcal{D}_{X \rightarrow Y} \otimes_{f^{-1}\mathcal{D}_Y} f^{-1}\mathcal{M} ,$$

taking account also of the structure as  $\mathcal{D}_X$ -module.

### 2.3 Comparison

Let  $\mathcal{M}$  be a  $\mathcal{D}_Y$ -module. We regard it as a connection on  $Y$  and consider its inverse image as a connection. The action of derivations is described in terms of local coordinates  $y_i$  on  $Y$ , by

$$\begin{aligned} (f^*\Delta)_\partial(\alpha \otimes m) &= \langle \partial, (f^*\nabla)(\alpha \otimes m) \rangle \\ &= \langle \partial, \alpha \nabla(m) + d(\alpha) \otimes m \rangle \\ &= \langle \partial, \alpha \left( \sum_i dy_i \otimes \Delta_{\eta_i}(m) \right) + d(\alpha) \otimes m \rangle \\ &= \alpha \left( \sum_i \partial(y_i) \otimes \Delta_{\eta_i}(m) \right) + \partial(\alpha) \otimes m \end{aligned}$$

where  $\partial$  is a section of  $\mathcal{D}er_X$ ,  $m$  is a section of  $\mathcal{M}$  (or  $f^{-1}\mathcal{M}$ ),  $\alpha$  is a section of  $\mathcal{O}_X$  (and  $\eta_i$  is the dual basis of  $dy_i$ ). Therefore the local descriptions make clear that for a connection  $\mathcal{E}$  on  $Y$ , the inverse image as a connection induces the structure of  $\mathcal{D}_X$ -module given by the inverse image of the corresponding  $\mathcal{D}_Y$ -module.

### §3 Direct image for connections and $\mathcal{D}$ -modules (and comparison following [DMSS])

Let  $f : X \rightarrow Y$  be a smooth morphism of smooth  $K$ -varieties. In order to compare the notions of (derived) direct images in the category of connections (the Gauss-Manin connections) and in the category of  $\mathcal{D}$ -modules, we need some preliminary materials, most concerning right  $\mathcal{D}$ -modules, De Rham functors, differential complexes (and the M.Saito equivalence). In this paper, we use the convention that the shift of complexes does not change the sign of differentials.

#### 3.1 Right and left $\mathcal{D}$ -modules

We denote by  $\mathcal{D}\text{-Mod}$  the category of left  $\mathcal{D}$ -modules and by  $\text{Mod-}\mathcal{D}$  the category of right  $\mathcal{D}$ -modules. It is well known that  $\omega_X = \Omega_X^{\dim X}$  has a canonical structure of right  $\mathcal{D}_X$ -module (see [Bo, VI, 3.2]). Let us define  $\omega_X(\mathcal{D}_X) = \omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X$ . It is endowed with two different structures of right  $\mathcal{D}_X$ -module: the first comes from the right multiplication on  $\mathcal{D}_X$  and the other is induced by the tensor product over  $\mathcal{O}_X$  of a right and a left  $\mathcal{D}_X$ -module. There exists a unique involution  $\iota : \omega_X(\mathcal{D}_X) \rightarrow \omega_X(\mathcal{D}_X)$  which is the identity on  $\omega_X$  and exchanges these two right  $\mathcal{D}_X$ -module structures (see [Sa, 1.7], using local coordinates  $x_i$  on  $X$ , the involution sends  $\omega \otimes P$  to  $\omega \otimes P^*$ , where  $\omega = dx_1 \wedge \cdots \wedge dx_{d_X}$ , and  $P^*$  is the transposition of  $P$ , defined by  $\alpha^* = \alpha$  for sections of  $\mathcal{O}_X$ ,  $\partial_i^* = -\partial_i$ , and  $(PQ)^* = Q^*P^*$ ). In the same way, we define  $\omega_X^{-1}(\mathcal{D}_X) = \mathcal{D}_X \otimes_{\mathcal{O}_X} \omega_X^{-1} = \mathcal{H}om_{\mathcal{D}_X}(\omega_X(\mathcal{D}_X), \mathcal{D}_X)$  and we notice that  $\omega_X^{-1}(\mathcal{D}_X)$  has two compatible and “interchangeable” structures of left  $\mathcal{D}_X$ -module.

We have an equivalence of categories between  $\mathcal{D}_X\text{-Mod}$  and  $\text{Mod-}\mathcal{D}_X$  given by the quasi inverse functors:

$$\begin{array}{ccc} \mathcal{D}_X\text{-Mod} & \longleftrightarrow & \text{Mod-}\mathcal{D}_X \\ \mathcal{M} & \longmapsto & \omega_X(\mathcal{M}) = \omega_X(\mathcal{D}_X) \otimes_{\mathcal{D}_X} \mathcal{M} \\ \mathcal{N} \otimes_{\mathcal{D}_X} \omega_X^{-1}(\mathcal{D}_X) = \omega_X^{-1}(\mathcal{N}) & \longleftarrow & \mathcal{N} . \end{array}$$

#### 3.2 De Rham functor for right and left $\mathcal{D}$ -modules

Let  $(\mathcal{E}, \nabla)$  be a connection on  $X$ . By definition its De Rham complex is  $\Omega_X^\bullet(\mathcal{E}) = \Omega_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{E}$  where the differentials are induced by the connection  $\nabla$  as usual:  $\nabla(\omega \otimes e) = d(\omega) \otimes e + (-)^{\deg \omega} \omega \wedge \nabla(e)$ .

The De Rham functor for left  $\mathcal{D}$ -modules is defined to be compatible with the notion of De Rham complex for connections, up to a shift. Let us consider  $\mathcal{D}_X$  as a left  $\mathcal{D}_X$ -module, then its De Rham complex as a connection is  $\Omega_X^\bullet(\mathcal{D}_X) = \Omega_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{D}_X$  (usual differentials). It is a resolution of  $\omega_X[-\dim X]$  in  $\text{Mod-}\mathcal{D}_X$ . For this reason, it is usual to define  $\text{DR}_X(\mathcal{D}_X) = \Omega_X^\bullet(\mathcal{D}_X)[\dim X]$ , so that  $\text{DR}_X(\mathcal{D}_X)$  is a resolution of  $\omega_X$  in  $\text{Mod-}\mathcal{D}_X$ .

Now, if  $\mathcal{M}$  is a left  $\mathcal{D}_X$ -module we define  $\text{DR}_X(\mathcal{M}) = \text{DR}_X(\mathcal{D}_X) \otimes_{\mathcal{D}_X} \mathcal{M}$  which is a complex of  $K_X$ -vector spaces. This functor extends to complexes and gives in the derived categories the functor  $\text{DR}_X : \mathbf{D}(\mathcal{D}_X\text{-Mod}) \rightarrow \mathbf{D}(K_X)$  (where  $\mathbf{D}(K_X)$  is the derived category of sheaves in  $K_X$ -vector spaces). Let us observe that for any  $\mathcal{M} \in \mathbf{D}(\mathcal{D}_X\text{-Mod})$  we have  $\text{DR}_X(\mathcal{M}) = \omega_X \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{M}$ .

The De Rham functor for right  $\mathcal{D}$ -modules is defined to be compatible with the left/right equivalence. Let us consider  $\mathcal{D}_X$  as a right  $\mathcal{D}_X$ -module, then its De Rham complex is  $\Theta_X^\bullet(\mathcal{D}_X) = \mathcal{D}_X \otimes_{\mathcal{O}_X} \Theta_X^\bullet$  (where  $\Theta_X^\bullet = \bigwedge^{\bullet} \mathcal{D}er_X$  and the differentials are locally defined by a Koszul complex). It is a resolution of  $\mathcal{O}_X$  in  $\mathcal{D}_X\text{-Mod}$ . Now, if  $\mathcal{N}$  is a right  $\mathcal{D}_X$ -module, then  $\text{DR}_X(\mathcal{N}) = \mathcal{N} \otimes_{\mathcal{D}_X} \Theta_X^\bullet(\mathcal{D}_X)$  as a functor  $\text{Mod-}\mathcal{D}_X \rightarrow \mathbf{C}(K_X)$ . The definition naturally extends to the category of complexes of  $\text{Mod-}\mathcal{D}_X$ , and to the derived category as before.

The compatibility between De Rham functors is expressed by the relations  $\text{DR}_X(\mathcal{M}) = \text{DR}_X(\omega_X(\mathcal{M}))$  and  $\text{DR}_X(\mathcal{N}) = \text{DR}_X(\omega_X^{-1}(\mathcal{N}))$ .

**3.2.1 Relative De Rham functor.** Let  $f : X \rightarrow Y$  be a smooth morphism between smooth  $K$ -varieties. The morphism  $f^*(\Omega_Y^1) \rightarrow \Omega_X^1$  induces a canonical short exact sequence

$$(3.2.2) \quad 0 \rightarrow f^*(\Omega_Y^1) \rightarrow \Omega_X^1 \rightarrow \Omega_{X/Y}^1 \rightarrow 0$$

where  $\Omega_{X/Y}^1$  is the sheaf of relative differential forms of degree one. Moreover any  $\mathcal{O}_X$ -module in (3.2.2) is locally free of finite type so (3.2.2) locally splits. Let  $\Theta_{X/Y} = \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/Y}^1, \mathcal{O}_X)$  be the  $\mathcal{O}_X$ -dual of  $\Omega_{X/Y}^1$  and let  $\mathcal{D}_{X/Y}$  denote the  $\mathcal{O}_X$ -algebra generated by  $\mathcal{O}_X$  and  $\Theta_{X/Y}$ . As in 1.1 for any  $\mathcal{O}_X$ -module  $\mathcal{E}$  the following supplementary structures on  $\mathcal{E}$  are equivalent:

- (1) an integrable relative connection, that is a morphism of  $f^{-1}(\mathcal{O}_Y)$ -modules  $\nabla_{X/Y} : \mathcal{E} \rightarrow \Omega_{X/Y}^1 \otimes_{\mathcal{O}_X} \mathcal{E}$  Leibniz with respect to sections of  $\mathcal{O}_X$  and such that  $\nabla_{X/Y}^2 = 0$  for the natural extension of  $\nabla_{X/Y}$ ;
- (2) a structure of left  $\mathcal{D}_{X/Y}$ -module on  $\mathcal{E}$ .

For any  $\mathcal{O}_X$ -module  $\mathcal{E}$  endowed with a relative integrable connection  $\nabla_{X/Y}$  let us define its relative De Rham complex as  $\Omega_{X/Y}^\bullet(\mathcal{E}) = \Omega_{X/Y}^\bullet \otimes_{\mathcal{O}_X} \mathcal{E}$  where the differentials are induced by  $\nabla_{X/Y}$ .

The relative De Rham functor for left  $\mathcal{D}_{X/Y}$ -modules is defined to be compatible with the De Rham functor for connection up to a shift (as in the case of  $\mathcal{D}_X$ -modules), and it induces a functor of derived categories:

$$\begin{aligned} \mathrm{DR}_{X/Y} : \mathbf{D}(\mathcal{D}_{X/Y}) &\longrightarrow \mathbf{D}(f^{-1}(\mathcal{O}_Y)) \\ \mathcal{E} &\longmapsto \mathrm{DR}_{X/Y}(\mathcal{E}) = \Omega_{X/Y}^\bullet(\mathcal{E})[d_{X/Y}] \end{aligned}$$

where  $d_{X/Y}$  is the relative dimension  $d_X - d_Y$ .

### 3.3 Direct images for connections (the Gauss-Manin connections)

The Leray filtration  $\mathrm{Ler}$  on  $\Omega_X^\bullet$  is defined by  $\mathrm{Ler}^p \Omega_X^\bullet = \mathrm{Im}(f^* \Omega_Y^p \otimes_{\mathcal{O}_X} \Omega_X^{\bullet-p} \rightarrow \Omega_X^\bullet)$  and, since  $f$  is a smooth morphism, the associated graded  $\mathcal{O}_X$ -module has  $\mathrm{Gr}_{\mathrm{Ler}}^p \Omega_X^\bullet \cong f^* \Omega_Y^p \otimes_{\mathcal{O}_X} \Omega_{X/Y}^{\bullet-p}$ .

If  $\mathcal{E}$  is a connection, we define the Leray filtration on its De Rham complex  $\Omega_X^\bullet(\mathcal{E})$  by the tensor product:  $\mathrm{Ler}^p \Omega_X^\bullet(\mathcal{E}) = \mathrm{Ler}^p \Omega_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{E}$ . Therefore the graded pieces are  $\mathrm{Gr}_{\mathrm{Ler}}^p \Omega_X^\bullet(\mathcal{E}) = f^* \Omega_Y^p \otimes_{\mathcal{O}_X} \Omega_{X/Y}^\bullet(\mathcal{E})[-p]$ .

The Leray filtration induces the Leray spectral sequence for the direct image functor by  $f$ :

$$(3.3.1) \quad E_1^{p,q} = \Omega_Y^p(R^{p+q} f_* \Omega_{X/Y}^\bullet(\mathcal{E})[-p]) \implies \mathbf{R}^n f_* \Omega_X^\bullet(\mathcal{E})$$

in the category of  $\mathcal{O}_Y$ -modules with differential operators (the complexes appearing in the spectral sequences are differential complexes on  $Y$ ). The differentials of  $E_1^{p,q}$  define the Gauss-Manin connections on the  $\mathcal{O}_Y$ -modules  $R^q f_* \Omega_{X/Y}^\bullet(\mathcal{E})$ , since

$$(3.3.2) \quad d_1^{p,q} : E_1^{p,q} = \Omega_Y^p(R^{p+q} f_* \Omega_{X/Y}^\bullet(\mathcal{E})[-p]) \longrightarrow E_1^{p+1,q} = \Omega_Y^{p+1}(R^{p+q+1} f_* \Omega_{X/Y}^\bullet(\mathcal{E})[-p-1])$$

(it is explicitly given by the connection homomorphism for the direct image functor of the short exact sequence of complexes

$$(3.3.3) \quad 0 \longrightarrow \mathrm{Gr}_{\mathrm{Ler}}^{p+1} \Omega_X^\bullet(\mathcal{E}) \longrightarrow \mathrm{Ler}^p \Omega_X^\bullet(\mathcal{E}) / \mathrm{Ler}^{p+2} \Omega_X^\bullet(\mathcal{E}) \longrightarrow \mathrm{Gr}_{\mathrm{Ler}}^p \Omega_X^\bullet(\mathcal{E}) \longrightarrow 0$$

which gives a piece of  $E_1$ ).

### 3.4 Direct images for $\mathcal{D}$ -modules

The direct image for  $\mathcal{D}$ -modules is defined using the following transfer modules:

- (3.4.1)  $\mathcal{D}_{X \rightarrow Y} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y$  which is in  $\mathcal{D}_X\text{-Mod-}f^{-1}\mathcal{D}_Y$  (the left  $\mathcal{D}_X$ -module structure is induced by (the tensor with) that of  $\mathcal{O}_X$ , the right  $f^{-1}\mathcal{D}_Y$ -module structure is induced by that of  $f^{-1}\mathcal{D}_Y$ , and the compatibility is obvious);
- (3.4.2)  $\mathcal{D}_{Y \leftarrow X} = \omega_X(\mathcal{D}_X) \otimes_{\mathcal{D}_X} \mathcal{D}_{X \rightarrow Y} \otimes_{f^{-1}\mathcal{D}_Y} f^{-1}\omega_Y^{-1}(\mathcal{D}_Y)$  which is in  $f^{-1}\mathcal{D}_Y\text{-Mod-}\mathcal{D}_X$  since it is obtained from  $\mathcal{D}_{X \rightarrow Y}$  by a double left/right exchange.

For a right  $\mathcal{D}_X$ -module  $\mathcal{N}$ , the direct image by  $f$  is defined by  $f_+ \mathcal{N} = \mathbf{R}f_*(\mathcal{N} \otimes_{\mathcal{D}_X}^{\mathbf{L}} \mathcal{D}_{X \rightarrow Y})$  as a right  $\mathcal{D}_Y$ -module. For a left  $\mathcal{D}_X$ -module  $\mathcal{M}$ , the direct image by  $f$  is defined by  $f_+ \mathcal{M} = \mathbf{R}f_*(\mathcal{D}_{Y \leftarrow X} \otimes_{\mathcal{D}_X}^{\mathbf{L}} \mathcal{M})$  as a left  $\mathcal{D}_Y$ -module. The compatibility of the two definitions is expressed by the following relations:

$$\omega_Y^{-1}(f_+(\mathcal{N})) = f_+(\omega_X^{-1}(\mathcal{N})) \quad \text{and} \quad \omega_Y(f_+(\mathcal{M})) = f_+(\omega_X(\mathcal{M})).$$

### 3.5 Differential complexes and M.Saito equivalence

Following M. Saito let  $\mathcal{O}_X\text{-Diff}$  be the category of  $\mathcal{O}_X$ -modules with differential operators as morphisms and let  $\mathbf{C}(\mathcal{O}_X\text{-Diff})$  be its category of complexes (see [DMSS]). Objects in  $\mathbf{C}(\mathcal{O}_X\text{-Diff})$  are called differential complexes on  $X$ . Any object in  $\mathbf{C}(\mathcal{O}_X\text{-Diff})$  could be regarded as a complex of  $K_X$ -vector spaces so that one has a functor  $F : \mathbf{C}(\mathcal{O}_X\text{-Diff}) \rightarrow \mathbf{C}(K_X)$ .

In [Sa,1.3.2] M.Saito defined the linearization functor  $\widetilde{\mathrm{DR}}_X^{-1} : \mathbf{C}(\mathcal{O}_X\text{-Diff}) \rightarrow \mathbf{C}(\mathrm{Mod}\text{-}\mathcal{D}_X)$  acting on the differential complex  $\mathcal{L}$  by  $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}_X$  (the differentials being extended to  $\mathcal{D}_X$ -linear maps). By localization with respect to the multiplicative system of quasi-isomorphisms on the right hand side and with respect to their pull-back on the left hand side (that is, the multiplicative system of  $\mathcal{D}_X$ -quasi-isomorphisms: morphisms of differential complexes whose linearization is a quasi-isomorphism in the category of right  $\mathcal{D}_X$ -modules) one obtains the functor  $\widetilde{\mathrm{DR}}_X^{-1} : \mathbf{D}(\mathcal{O}_X\text{-Diff}) \rightarrow \mathbf{D}(\mathrm{Mod}\text{-}\mathcal{D}_X)$ .

It is clear that for any right (resp. left)  $\mathcal{D}_X$ -module  $\mathcal{N}$  (resp.  $\mathcal{M}$ ), its De Rham complex is an object in  $\mathbf{C}(\mathcal{O}_X\text{-Diff})$ . In particular the functor  $\mathrm{DR}_X$  factors through  $\mathbf{C}(\mathcal{O}_X\text{-Diff})$  and we denote this factorization by  $\widetilde{\mathrm{DR}}_X : \mathbf{C}(\mathrm{Mod}\text{-}\mathcal{D}_X) \rightarrow \mathbf{C}(\mathcal{O}_X\text{-Diff})$ . In particular we have the following commutative diagram of functors

$$\begin{array}{ccc} \mathbf{C}(\mathrm{Mod}\text{-}\mathcal{D}_X) & \xrightarrow{\widetilde{\mathrm{DR}}_X} & \mathbf{C}(\mathcal{O}_X\text{-Diff}) \\ & \searrow \mathrm{DR}_X & \downarrow F \\ & & \mathbf{C}(K_X) \end{array}$$

M.Saito also proved that  $\widetilde{\mathrm{DR}}_X$  localizes with respect to quasi-isomorphisms on the left hand side and with respect to  $\mathcal{D}_X$ -quasi-isomorphisms on the right hand side, so it induces a functor  $\widetilde{\mathrm{DR}}_X : \mathbf{D}(\mathrm{Mod}\text{-}\mathcal{D}_X) \rightarrow \mathbf{D}(\mathcal{O}_X\text{-Diff})$ .

**3.5.1 Proposition (M.Saito's equivalence).** *The functor  $\widetilde{\mathrm{DR}}_X^{-1}$  is an equivalence of categories whose quasi-inverse is  $\widetilde{\mathrm{DR}}_X$ . In particular we have canonical quasi-isomorphisms*

$$\widetilde{\mathrm{DR}}_X^{-1} \widetilde{\mathrm{DR}}_X(\mathcal{N}) \cong \mathcal{N} \quad \text{and} \quad \widetilde{\mathrm{DR}}_X^{-1} \widetilde{\mathrm{DR}}_X(\mathcal{M}) \cong \omega_X(\mathcal{M}),$$

both in  $\mathbf{C}(\mathrm{Mod}\text{-}\mathcal{D}_X)$  (see [Sa,1.8]).

**3.5.2** In the case of right  $\mathcal{D}_X$ -modules, there is also a compatibility with the direct image of differential complexes (which is induced by the usual direct image for abelian sheaves), via the linearization functor:  $\widetilde{\mathrm{DR}}_Y^{-1} \mathbf{R}f_*(\mathcal{L}) = f_+ \widetilde{\mathrm{DR}}_X^{-1}(\mathcal{L})$  (see [DMSS,1.3.2]).

**3.6. Theorem (comparison for direct images, following [DMSS]).** *Let  $f : X \rightarrow Y$  be a smooth morphism of smooth  $K$ -varieties. For any left  $\mathcal{D}_X$ -module  $\mathcal{M}$  (identified with a connection on  $X$ ) and for any  $q$  we have natural isomorphisms  $R^q f_* \mathrm{DR}_{X/Y}(\mathcal{M}) \cong \mathcal{H}^q(f_+ \mathcal{M})$  in the category of left  $\mathcal{D}_Y$ -modules, where the left hand side has the structure of the Gauss-Manin connection.*

**Proof.** Let consider the Leray spectral sequence  $E$  of  $\mathcal{M}$  with respect to  $f$ . Since  $\mathcal{D}_Y$  is a flat  $\mathcal{O}_Y$ -module, we may apply the linearization functor  $\widetilde{\mathrm{DR}}_Y^{-1}$  to obtain the spectral sequence

$$\widetilde{\mathrm{DR}}_Y^{-1} E_1^{p,q} = \widetilde{\mathrm{DR}}_Y^{-1} \Omega_Y^p(R^{p+q} f_* \Omega_{X/Y}^\bullet(\mathcal{M})[-p]) \implies \widetilde{\mathrm{DR}}_Y^{-1} R^n f_* \Omega_X^\bullet(\mathcal{M})$$

in the category of right  $\mathcal{D}_Y$ -modules. Now (by 3.5.1) the complex  $\widetilde{\mathrm{DR}}_Y^{-1} E_1^{\bullet,q}$  is quasi-isomorphic to

$$\widetilde{\mathrm{DR}}_Y^{-1} \widetilde{\mathrm{DR}}_Y[-\dim Y](R^q f_* \mathrm{DR}_{X/Y}[-d_{X/Y}](\mathcal{M})) \cong \omega_Y(R^q f_* \mathrm{DR}_{X/Y}(\mathcal{M}))[-\dim Y - d_{X/Y}]$$

(where  $d_{X/Y} = \dim X - \dim Y$  is the relative dimension) so that the spectral sequence degenerates at  $E_2$ ; while the limit is quasi-isomorphic (by 3.5.2, 3.5.1 and 3.4) to

$$\begin{aligned} \widetilde{\mathrm{DR}}_Y^{-1} \mathbf{R}f_* \widetilde{\mathrm{DR}}_X[-\dim X](\mathcal{M}) &\cong f_+(\widetilde{\mathrm{DR}}_X^{-1} \widetilde{\mathrm{DR}}_X[-\dim X](\mathcal{M})) \\ &\cong f_+(\omega_X(\mathcal{M})[-\dim X]) \\ &\cong \omega_Y(f_+(\mathcal{M})[-\dim X]). \end{aligned}$$

So we have the isomorphisms  $R^q f_* \mathrm{DR}_{X/Y}(\mathcal{M}) \cong \mathcal{H}^{q+\dim X}(f_+(\mathcal{M})[-\dim X])$  in the category of complexes of left  $\mathcal{D}_Y$ -modules, from which the proposition follows.  $\square$

#### §4. Alternative proof of the comparison between direct images

We present here an alternative proof of the comparison theorem for direct images, which is in some sense more elementary, since Saito's equivalence is not used. The strategy we discuss here also has the advantage of clarifying the structure of the Gauss-Manin connection, taking account of one of its avatars before the application of the derived direct image functor. In fact we compare two structures of left  $f^{-1}\mathcal{D}_Y$ -modules, one defining the Gauss-Manin connection, the other coming from the structure of the transfer module  $\mathcal{D}_{Y \leftarrow X}$ . The main technical tool is the commutativity of a diagram in the derived category of right  $\mathcal{D}_X$ -modules (see Proposition 4.3.2), for which the homotopy lemma 4.3.1 is used.

##### 4.1 The distinguished triangle for the Gauss-Manin connection

Let us consider the exact sequence (3.3.3) for  $p = 0$  and  $\mathcal{E} = \mathcal{D}_X$  (in the category of complexes of right  $\mathcal{D}_X$ -modules):

$$(4.1.0) \quad 0 \longrightarrow f^*(\Omega_Y^1) \otimes_{\mathcal{O}_X} \Omega_{X/Y}^\bullet(\mathcal{D}_X)[-1] \xrightarrow{i} \mathrm{Ler}^0 \Omega_X^\bullet(\mathcal{D}_X) / \mathrm{Ler}^2 \Omega_X^\bullet(\mathcal{D}_X) \xrightarrow{\pi} \Omega_{X/Y}^\bullet(\mathcal{D}_X) \longrightarrow 0$$

(recall that  $\mathrm{Gr}_{\mathrm{Ler}}^1(\Omega_X^\bullet(\mathcal{D}_X)) \cong f^*(\Omega_Y^1) \otimes_{\mathcal{O}_X} \Omega_{X/Y}^\bullet(\mathcal{D}_X)[-1]$  and  $\mathrm{Gr}_{\mathrm{Ler}}^0(\Omega_X^\bullet(\mathcal{D}_X)) \cong \Omega_{X/Y}^\bullet(\mathcal{D}_X)$ ) which gives the distinguished triangle:

$$(4.1.1) \quad \begin{array}{ccc} f^*(\Omega_Y^1) \otimes_{\mathcal{O}_X} \Omega_{X/Y}^\bullet(\mathcal{D}_X)[-1] & \xrightarrow{i} & \mathrm{Ler}^0 \Omega_X^\bullet(\mathcal{D}_X) / \mathrm{Ler}^2 \Omega_X^\bullet(\mathcal{D}_X) \\ & \searrow \begin{smallmatrix} [+1] \\ \delta(\mathcal{D}_X) \end{smallmatrix} & \downarrow \pi \\ & & \Omega_{X/Y}^\bullet(\mathcal{D}_X) \end{array}$$

in  $\mathbf{D}^b(\mathrm{Mod}\text{-}\mathcal{D}_X)$  (derived category of right  $\mathcal{D}_X$ -modules). The connecting morphism  $\delta(\mathcal{D}_X)$  is represented in the derived category by the following diagram (in which the mapping cone of  $i$  appears):

$$(4.1.2) \quad \begin{array}{ccc} (f^*(\Omega_Y^1) \otimes_{\mathcal{O}_X} \Omega_{X/Y}^\bullet(\mathcal{D}_X)) \oplus \mathrm{Ler}^0 \Omega_X^\bullet(\mathcal{D}_X) / \mathrm{Ler}^2 \Omega_X^\bullet(\mathcal{D}_X) & & \\ \mathrm{qis} \downarrow (0, \pi) & \searrow -p_1 = (-1, 0) & \\ \Omega_{X/Y}^\bullet(\mathcal{D}_X) & & f^*(\Omega_Y^1) \otimes_{\mathcal{O}_X} \Omega_{X/Y}^\bullet(\mathcal{D}_X) \end{array}$$

The fundamental fact here is that *the Gauss-Manin connection comes by applying the derived functor  $\mathbf{R}f_*$  to this connecting morphism.*

Notice moreover that for any left  $\mathcal{D}_X$ -module  $\mathcal{E}$  we can apply the derived functor  $-\otimes_{\mathcal{D}_X}^{\mathbf{L}} \mathcal{E}$  to the distinguished triangle (4.1.1) so we obtain the distinguished triangle:

$$\begin{array}{ccc} f^*(\Omega_Y^1) \otimes_{\mathcal{O}_X} \Omega_{X/Y}^\bullet(\mathcal{E})[-1] & \longrightarrow & \mathrm{Ler}^0 \Omega_X^\bullet(\mathcal{E}) / \mathrm{Ler}^2 \Omega_X^\bullet(\mathcal{E}) \\ & \searrow \begin{smallmatrix} [+1] \\ \delta(\mathcal{E}) \end{smallmatrix} & \downarrow \\ & & \Omega_{X/Y}^\bullet(\mathcal{E}) \end{array}$$

(really, we do not need to take the derived functor  $-\otimes_{\mathcal{D}_X}^{\mathbf{L}} \mathcal{E}$  but simply  $-\otimes_{\mathcal{D}_X} \mathcal{E}$  because any complex in (4.1.1) is acyclic for  $-\otimes_{\mathcal{D}_X} \mathcal{E}$ ). This is the distinguished triangle induced by the short exact sequence (3.3.3) for  $p = 0$ .

##### 4.2 The connection associated to $\mathcal{D}_{Y \leftarrow X}$

In view of the comparison, we have to give a concrete expression in terms of a connection (in the derived category) for the  $\mathcal{D}_Y$ -module structure of the derived direct image.

**4.2.1 Lemma [Lau, 5.2.3.4].** *Let  $f : X \rightarrow Y$  be a smooth morphism of smooth  $K$ -varieties. There exists a canonical morphism of complexes of right  $\mathcal{D}_X$ -modules  $\lambda : \mathrm{DR}_{X/Y}(\mathcal{D}_X) \rightarrow \mathcal{D}_{Y \leftarrow X}$  which is a quasi-isomorphism. In particular,  $\mathrm{DR}_{X/Y}(\mathcal{D}_X)$  is a left resolution of  $\mathcal{D}_{Y \leftarrow X}$  in the category  $f^{-1}\mathcal{O}_Y\text{-Mod-}\mathcal{D}_X$ , with locally free right  $\mathcal{D}_X$ -modules. As a consequence, the complex  $\mathrm{DR}_{X/Y}(\mathcal{D}_X)$  admits a structure of left  $f^{-1}\mathcal{D}_Y$ -module, induced by transfer of structure via the quasi-isomorphism, in the derived category of  $f^{-1}\mathcal{O}_Y\text{-Mod}$ .*

**Proof.** The canonical morphism  $\lambda$  is defined by the composition

$$\omega_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{D}_X \xrightarrow{i} \mathcal{D}_{Y \leftarrow X} \otimes_{\mathcal{O}_X} \mathcal{D}_X \xrightarrow{\cdot} \mathcal{D}_{Y \leftarrow X}$$

where the first map comes from the canonical inclusion of  $\omega_{X/Y}$  into  $\mathcal{D}_{Y \leftarrow X} = \omega_{X/Y} \otimes_{\mathcal{O}_X} f^*\mathcal{D}_Y$ , and the second one uses the canonical structure of right  $\mathcal{D}_X$ -module of  $\mathcal{D}_{Y \leftarrow X}$ . A local computation using the canonical filtrations by the order of differential operators shows that the graded pieces are Koszul complexes, so that the assertion follows (see [Lau, 5.2.3.4] for details).  $\square$

Notice moreover that the morphism  $\lambda$  may be described in terms of the canonical map  $\mathcal{D}_f : \mathcal{D}_X \rightarrow f^*\mathcal{D}_Y$  composing  $\mathrm{id} \otimes \mathcal{D}_f$  with the transposition of differential operators, since the following diagram

$$\begin{array}{ccc} \omega_{X/Y} \otimes \mathcal{D}_X \cong \mathcal{H}om(f^{-1}\omega_Y, \omega_X \otimes \mathcal{D}_X) & \xrightarrow{i} & \mathcal{H}om(f^{-1}\omega_Y, (\omega_X \otimes f^*\mathcal{D}_Y) \otimes \mathcal{D}_X) \cong \mathcal{D}_{Y \leftarrow X} \otimes \mathcal{D}_X \\ \downarrow \iota & & \downarrow \cdot \\ \mathcal{H}om(f^{-1}\omega_Y, \omega_X \otimes \mathcal{D}_X) & \xrightarrow{\mathcal{D}_f} & \mathcal{H}om(f^{-1}\omega_Y, \omega_X \otimes f^*\mathcal{D}_Y) \cong \mathcal{D}_{Y \leftarrow X} \end{array}$$

commutes (here,  $\mathcal{H}om$  means  $\mathcal{H}om_{f^{-1}\mathcal{O}_Y}$ , the symbol  $\nabla$  indicates the  $f^{-1}\mathcal{O}_Y$ -module structure used in the  $\mathcal{H}om$ ,  $\otimes$  means  $\otimes_{\mathcal{O}_X}$ , and  $\iota$  is the canonical involution of  $\omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X$  exchanging the two structures of right  $\mathcal{D}_X$ -module).

**4.2.2** Notice that the  $f^{-1}\mathcal{D}_Y$ -module structure of  $\mathrm{DR}_{X/Y}(\mathcal{D}_X)$  is described in terms of its connection by the following diagram

$$\begin{array}{ccc} \mathrm{DR}_{X/Y}(\mathcal{D}_X) & & f^{-1}\Omega_Y^1 \otimes_{f^{-1}\mathcal{O}_Y} \mathrm{DR}_{X/Y}(\mathcal{D}_X) \\ \text{qis} \downarrow \lambda & & \text{qis} \downarrow 1 \otimes \lambda \\ \mathcal{D}_{Y \leftarrow X} & \xrightarrow{\nabla_{Y \leftarrow X}} & f^{-1}\Omega_Y^1 \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{D}_{Y \leftarrow X} \end{array}$$

where  $\nabla_{Y \leftarrow X}$  is the connection (Leibniz with respect to section of  $f^{-1}(\mathcal{O}_Y)$ ) induced by the  $f^{-1}(\mathcal{D}_Y)$ -module structure of  $\mathcal{D}_{Y \leftarrow X}$ .

**4.2.3 Corollary.** *Let  $f : X \rightarrow Y$  be a smooth morphism of smooth  $K$ -varieties. For any left  $\mathcal{D}_X$ -module  $\mathcal{M}$  there is a canonical quasi-isomorphism  $\lambda(\mathcal{M}) : \mathrm{DR}_{X/Y}(\mathcal{M}) \rightarrow \mathcal{D}_{Y \leftarrow X} \otimes_{\mathcal{D}_X}^{\mathbf{L}} \mathcal{M}$ , so that  $\mathrm{DR}_{X/Y}(\mathcal{M})$  admits a structure of left  $f^{-1}(\mathcal{D}_Y)$ -module in the derived category of  $f^{-1}(\mathcal{O}_Y)$ -modules.*

Applying the derived direct image functor to the above morphism, one obtains a canonical quasi-isomorphism  $\mathbf{R}f_*\mathrm{DR}_{X/Y}(\mathcal{M}) \rightarrow f_+(\mathcal{M})$  in the category of complexes of  $\mathcal{O}_Y$ -modules, so that the complex  $\mathbf{R}f_*\mathrm{DR}_{X/Y}(\mathcal{M})$  admits a structure of left  $\mathcal{D}_Y$ -module in the derived category of  $\mathcal{O}_Y$ -modules.

**4.3** The following two results are the kernel of our comparison argument; essentially we have to compare the diagram (4.1.2) with the diagram in 4.2.2. To do that, we re-write the projection  $p_1$  of (4.1.2) up to homotopy.

**4.3.1 (homotopy) Lemma.** *Let  $i : f^*(\Omega_Y^1) \otimes_{\mathcal{O}_X} \Omega_{X/Y}^\bullet(\mathcal{D}_X)[-1] \rightarrow \mathrm{Ler}^0 \Omega_X^\bullet(\mathcal{D}_X) / \mathrm{Ler}^2 \Omega_X^\bullet(\mathcal{D}_X)$  be the canonical inclusion.*

- (i) the identity morphism of the mapping cone of  $i$  is homotopic to the morphism  $\Psi^\bullet$  defined by  $\Psi^{d_{X/Y}} = \begin{pmatrix} 0 & -\phi^{-1}d \\ 0 & 1 \end{pmatrix}$  and  $\Psi^q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  for  $q \neq d_{X/Y}$ ;  
(ii) the connecting morphism

$$-p_1 = (-1, 0) : (f^*(\Omega_Y^1) \otimes_{\mathcal{O}_X} \Omega_{X/Y}^\bullet(\mathcal{D}_X)) \oplus \text{Ler}^0 \Omega_X^\bullet(\mathcal{D}_X) / \text{Ler}^2 \Omega_X^\bullet(\mathcal{D}_X) \rightarrow (f^*(\Omega_Y^1) \otimes_{\mathcal{O}_X} \Omega_{X/Y}^\bullet(\mathcal{D}_X))$$

of the distinguished triangle generated by  $i$  is homotopic to the morphism  $\psi^\bullet = -p_1 \circ \Psi^\bullet$ , that is  $\psi^{d_{X/Y}} = (0, \phi^{-1}d)$  and  $\psi^q = (-1, 0)$  for  $q \neq d_{X/Y}$ ;

where  $\phi$  is the canonical isomorphism  $f^* \Omega_Y^1 \otimes_{\mathcal{O}_X} \omega_{X/Y}(\mathcal{D}_X) \longrightarrow \frac{\text{Ler}^0}{\text{Ler}^2}(\Omega_X^{d_{X/Y}+1}(\mathcal{D}_X))$  induced by  $i$ .

**Proof.** Let us consider the exact sequence of complexes (4.1.0) in degrees  $d_{X/Y} - 1, d_{X/Y}, d_{X/Y} + 1$ :

$$\begin{array}{ccccccc} 0 & \longrightarrow & f^* \Omega_Y^1 \otimes_{\mathcal{O}_X} \Omega_{X/Y}^{d_{X/Y}-2}(\mathcal{D}_X) & \xrightarrow{i} & \frac{\text{Ler}^0}{\text{Ler}^2}(\Omega_X^{d_{X/Y}-1}(\mathcal{D}_X)) & \xrightarrow{\pi} & \Omega_{X/Y}^{d_{X/Y}-1}(\mathcal{D}_X) \longrightarrow 0 \\ & & \downarrow d & & \downarrow d & & \downarrow d \\ 0 & \longrightarrow & f^* \Omega_Y^1 \otimes_{\mathcal{O}_X} \Omega_{X/Y}^{d_{X/Y}-1}(\mathcal{D}_X) & \xrightarrow{i} & \frac{\text{Ler}^0}{\text{Ler}^2}(\Omega_X^{d_{X/Y}}(\mathcal{D}_X)) & \xrightarrow{\pi} & \omega_{X/Y}(\mathcal{D}_X) \longrightarrow 0 \\ & & \downarrow d & & \downarrow d & & \downarrow \\ 0 & \longrightarrow & f^* \Omega_Y^1 \otimes_{\mathcal{O}_X} \omega_{X/Y}(\mathcal{D}_X) & \xrightarrow[\cong]{\phi} & \frac{\text{Ler}^0}{\text{Ler}^2}(\Omega_X^{d_{X/Y}+1}(\mathcal{D}_X)) & \longrightarrow & 0 \longrightarrow 0 \end{array}$$

from which one deduces that  $\phi$  is an isomorphism.

- (i) Using the homotopy map of the mapping cone of  $i$  which is zero for degrees different from  $d_{X/Y} + 1$ , and  $\begin{pmatrix} -\phi^{-1} \\ 0 \end{pmatrix}$  in degree  $d_{X/Y} + 1$ , we have that the identity map of the mapping cone is homotopic to the morphism having the following expression in degree  $d_{X/Y}$ :  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -\phi^{-1} \\ 0 \end{pmatrix}(\phi, d) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & -\phi^{-1}d \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\phi^{-1}d \\ 0 & 1 \end{pmatrix}$  and unchanged otherwise, as stated.  
(ii) This follows from the previous point, since  $\text{id} \sim \Psi^\bullet$  implies  $p_1 \sim p_1 \circ \Psi^\bullet = \psi^\bullet$ . Explicitly, we may use the homotopy map of the connecting morphism which is zero for degrees different from  $d_{X/Y} + 1$ , and  $\phi^{-1}$  in degree  $d_{X/Y} + 1$ . Then we have that  $-p_1$  is homotopic to the morphism having the following expression in degree  $d_{X/Y}$ :  $(-1, 0) + (\phi^{-1})(\phi, d) = (0, \phi^{-1}d)$  and unchanged otherwise, as stated.  $\square$

**4.3.2 Proposition.** The following diagram

$$\begin{array}{ccc} (f^*(\Omega_Y^1) \otimes_{\mathcal{O}_X} \text{DR}_{X/Y}(\mathcal{D}_X)) \oplus \text{Ler}^0 \text{DR}_X(\mathcal{D}_X) / \text{Ler}^2 \text{DR}_X(\mathcal{D}_X)[-d_Y] & & \\ \text{qis} \downarrow (0, \pi) & \searrow -p_1 = (-1, 0) & \\ \text{DR}_{X/Y}(\mathcal{D}_X) & & f^*(\Omega_Y^1) \otimes_{\mathcal{O}_X} \text{DR}_{X/Y}(\mathcal{D}_X) \\ \text{qis} \downarrow \lambda & & \text{qis} \downarrow 1 \otimes \lambda \\ \mathcal{D}_Y \leftarrow X & \xrightarrow{\nabla_Y \leftarrow X} & f^*(\Omega_Y^1) \otimes_{\mathcal{O}_X} \mathcal{D}_Y \leftarrow X \end{array}$$

commutes in the derived category of right  $\mathcal{D}_X$ -modules.

Since this diagram is just the superposition of (4.1.2) and 4.2.2, its commutativity identifies the two structures of left  $f^{-1}\mathcal{D}_Y$  module of  $\text{DR}_{X/Y}(\mathcal{D}_X)$ . Moreover, the result extends immediately to any left  $\mathcal{D}_X$ -module  $\mathcal{M}$ , by the tensor product over  $\mathcal{D}_X$ .

**Proof.** By the homotopy lemma, we may use  $\psi$  instead of  $-p_1$ , and since any object of the last row is



a complex concentrated in degree zero, we need only prove the commutativity of the following diagram:

$$\begin{array}{ccc}
\frac{\text{Ler}^0}{\text{Ler}^2}(\Omega_X^{d_{X/Y}}(\mathcal{D}_X)) & & \\
\downarrow \pi & \searrow \phi^{-1}d & \\
\omega_{X/Y}(\mathcal{D}_X) & & f^*(\Omega_Y^1) \otimes_{\mathcal{O}_X} \omega_{X/Y}(\mathcal{D}_X) \\
\downarrow \lambda & & \downarrow 1 \otimes \lambda \\
\mathcal{D}_{Y \leftarrow X} & \xrightarrow{\nabla_{Y \leftarrow X}} & f^*(\Omega_Y^1) \otimes_{\mathcal{O}_X} \mathcal{D}_{Y \leftarrow X}
\end{array}$$

(the first object of the mapping cone does not appear, because it is sent to zero using both morphisms).

Notice, first of all, that there exists a unique morphism  $\alpha : \omega_{X/Y}(\mathcal{D}_X) \rightarrow f^*\Omega_Y^1 \otimes_{\mathcal{O}_X} \mathcal{D}_{Y \leftarrow X}$  making the upper part (“parallelogram”) of the diagram commutative, since (with reference to the diagram in 4.3.1) we have that  $(1 \otimes \lambda)\phi^{-1}d = (1 \otimes \lambda)d = 0$ , so that  $(1 \otimes \lambda)\phi^{-1}d$  factors as  $\alpha\pi$ . As a consequence, we have to prove that  $\nabla_{Y \leftarrow X}\lambda = \alpha$ . One has two different possibilities for the proof. The first one is a local computation. Using local coordinates  $x_i$  ( $i = 1, \dots, d_X$ ) on  $X$  such that  $dx_1, \dots, dx_{d_{X/Y}}$  are generators of the relative differentials, and  $\omega = dx_1 \wedge \dots \wedge dx_{d_{X/Y}}$ , for the morphism  $\alpha$  we have the following expression

$$\begin{aligned}
\alpha(\omega \otimes \partial) &= (1 \otimes \lambda)\phi^{-1}d([\omega \otimes \partial]) \\
&= (1 \otimes \lambda)\phi^{-1} \left[ \sum_i dy_i \wedge \omega \otimes \eta_i \partial \right] \\
&= (1 \otimes \lambda) \left( \sum_i dy_i \otimes \omega \otimes \eta_i \partial \right) \\
&= \sum_i dy_i \otimes \omega \otimes (\eta_i \partial)^*
\end{aligned}$$

where local coordinates  $y_i$  on  $Y$  are used ( $dy_i$  and  $\eta_i$  are dual bases for  $\Omega_Y^1$  and  $\Theta_Y$ ); and  $\partial$  is a local section of  $\mathcal{D}_X$ . Therefore, the action of the derivative  $\eta_i$  (dual bases of  $dy_i$ ) is given by  $\omega \otimes (\eta_i \partial)^*$ . On the other hand, the structure of  $f^*\mathcal{D}_Y$ -module of  $\mathcal{D}_{Y \leftarrow X}$  is given by the twist of the right structure (by multiplication) of  $f^*\mathcal{D}_Y$  with the right structure of  $f^*\omega_Y$  (by the action of  $-\text{Lie}_\eta$ , which is trivial on  $\omega$ ), and has therefore the local expression  $\eta_i(\omega \otimes \partial) = -\omega \otimes \partial \eta_i$ ; composing with  $\lambda$ , the action of  $\eta_i$  sends  $\omega \otimes \partial$  to  $\omega \otimes \partial^* \eta_i^*$  and coincides with that given above.

The second possibility is more abstract and relies on the compatibility of De Rham functors. Since the morphism  $\alpha$  is induced by the differential of the absolute De Rham complex of  $\mathcal{D}_X$ , and it factors uniquely through  $\lambda$  (see again the diagram in 4.3.1, since  $(\pi)d\alpha = 0$  and  $\lambda$  gives a cokernel for  $d$  in the last column), it is sufficient to prove that the morphism  $\nabla_{Y \leftarrow X}$  is compatible with that complex. We observe that  $\omega_X(f^*\mathcal{D}_Y)$  has two compatible structures, as a right  $\mathcal{D}_X$ -module and as a right  $f^{-1}(\mathcal{D}_Y)$ -module. We define  $\text{DR}_Y(\omega_X(f^*\mathcal{D}_Y)) := \omega_X(f^*\mathcal{D}_Y) \otimes_{\mathcal{O}_X} \bigwedge^{\bullet} f^*\Theta_Y^1$  (the De Rham complex as right  $f^{-1}\mathcal{D}_Y$ -module) which is isomorphic (by left/right exchange on the  $f^{-1}(\mathcal{D}_Y)$ -module structure) to  $\text{DR}_Y(\omega_{X/Y}(f^*\mathcal{D}_Y)) := f^{-1}\Omega_Y^{\bullet} \otimes_{f^{-1}(\mathcal{O}_Y)} \omega_{X/Y}(f^*\mathcal{D}_Y)[d_Y]$ . Now, we have the following canonical morphisms of De Rham complexes:

$$\text{DR}_X(\mathcal{D}_X) \cong \text{DR}_X(\omega_X(\mathcal{D}_X)) \xrightarrow{\iota} \text{DR}_X(\omega_X(\mathcal{D}_X)) \longrightarrow \text{DR}_Y(\omega_X(f^*\mathcal{D}_Y)) \cong \text{DR}_Y(\omega_{X/Y}(f^*\mathcal{D}_Y))$$

where the isomorphisms  $\cong$  are given by left/right exchanges, the first morphism comes from the involution  $\iota$ , the second comes from  $\mathcal{D}_f : \mathcal{D}_X \rightarrow f^*\mathcal{D}_Y$ . In degrees  $-d_Y$  and  $-d_Y + 1$  we can read the following compatibilities:

$$\begin{array}{ccccccc}
\Omega_X^{d_{X/Y}}(\mathcal{D}_X) & \xrightarrow{\sim} & \omega_X(\mathcal{D}_X) \otimes \bigwedge^{d_Y} \Theta_X^1 & \rightarrow & \omega_X(f^*\mathcal{D}_Y) \otimes \bigwedge^{d_Y} f^*\Theta_Y^1 & \cong & \omega_{X/Y}(f^*\mathcal{D}_Y) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \nabla_{Y \leftarrow X} \\
\Omega_X^{d_{X/Y}+1}(\mathcal{D}_X) & \xrightarrow{\sim} & \omega_X(\mathcal{D}_X) \otimes \bigwedge^{d_Y-1} \Theta_X^1 & \rightarrow & \omega_X(f^*\mathcal{D}_Y) \otimes \bigwedge^{d_Y-1} f^*\Theta_Y^1 & \cong & f^{-1}\Omega_Y^1 \otimes \omega_{X/Y}(f^*\mathcal{D}_Y)
\end{array}$$

from which the result follows.  $\square$

**4.4 Theorem.** For any left  $\mathcal{D}_X$ -module  $\mathcal{M}$ , the canonical quasi-isomorphism

$$\mathbf{R}f_* \mathrm{DR}_{X/Y}(\mathcal{M}) \rightarrow \mathbf{R}f_*(\mathcal{D}_{Y \leftarrow X} \otimes_{\mathcal{D}_X}^{\mathbf{L}} \mathcal{M}) = f_+(\mathcal{M})$$

identifies the structures of  $\mathcal{D}_Y$ -modules of the two terms (the Gauss-Manin structure on the left hand side, and the canonical one on the right hand side), so that for any  $q$  we have natural isomorphisms

$$R^q f_* \mathrm{DR}_{X/Y}(\mathcal{M}) \cong \mathcal{H}^q(f_+ \mathcal{M})$$

in the category of left  $\mathcal{D}_Y$ -modules, where the left side has the structure of the Gauss-Manin connection.

**Proof.** We have (by lemma 4.2.3) a canonical quasi-isomorphism  $\mathrm{DR}_{X/Y}(\mathcal{M}) \rightarrow \mathcal{D}_{Y \leftarrow X} \otimes_{\mathcal{D}_X}^{\mathbf{L}} \mathcal{M}$  which identifies (by Proposition 4.3.2) the structures of  $f^{-1}\mathcal{D}_Y$ -modules of the two terms. Applying the derived functor  $\mathbf{R}f_*$  one obtains a canonical quasi-isomorphism  $\mathbf{R}f_* \mathrm{DR}_{X/Y}(\mathcal{M}) \rightarrow \mathbf{R}f_*(\mathcal{D}_{Y \leftarrow X} \otimes_{\mathcal{D}_X}^{\mathbf{L}} \mathcal{M}) = f_+(\mathcal{M})$  which identifies the structures of  $\mathcal{D}_Y$ -modules of the two terms. (Recall that  $\mathbf{R}f_*$  is intended as a functor from  $\mathbf{D}^+(f^{-1}\mathcal{D}_Y\text{-Mod})$  to  $\mathbf{D}^+(\mathcal{D}_Y\text{-Mod})$ .)  $\square$

**4.5** Finally, we translate the result in term of De Rham complexes.

**4.5.1 Lemma [Me ch.I, 5.4.3, 4, 4b].** Let  $f : X \rightarrow Y$  be a smooth morphism of smooth  $K$ -varieties. For any  $\mathcal{D}_X$ -module  $\mathcal{M}$  there exists a canonical quasi-isomorphism  $\mathrm{DR}_Y(f_+ \mathcal{M}) \rightarrow \mathbf{R}f_*(\mathrm{DR}_X(\mathcal{M}))$ , where  $\mathbf{R}f_*$  is the usual derived direct image for abelian sheaves.

**Proof.** The canonical morphism in the derived category of abelian sheaves on  $Y$

$$\omega_Y \otimes_{\mathcal{D}_Y}^{\mathbf{L}} \mathbf{R}f_*(\mathcal{D}_{Y \leftarrow X} \otimes_{\mathcal{D}_X}^{\mathbf{L}} \mathcal{M}) \longrightarrow \mathbf{R}f_*(f^{-1}\omega_Y \otimes_{f^{-1}\mathcal{D}_Y}^{\mathbf{L}} (\mathcal{D}_{Y \leftarrow X} \otimes_{\mathcal{D}_X}^{\mathbf{L}} \mathcal{M}))$$

is an isomorphism: this can be verified using a locally free resolution of  $\omega_Y$  in the category of right- $\mathcal{D}_Y$ -modules. Now the left hand side is canonically quasi-isomorphic to  $\omega_Y \otimes_{\mathcal{D}_Y}^{\mathbf{L}} f_+(\mathcal{M}) \cong \mathrm{DR}_Y(f_+ \mathcal{M})$  and the right hand side to  $\mathbf{R}f_*(\omega_X \otimes_{\mathcal{D}_X}^{\mathbf{L}} \mathcal{M}) \cong \mathbf{R}f_*(\mathrm{DR}_X(\mathcal{M}))$ .  $\square$

**4.5.2 Corollary.** Let  $f : X \rightarrow Y$  be a smooth morphism of smooth  $K$ -varieties. For any  $\mathcal{D}_X$ -module  $\mathcal{M}$  there exists a canonical quasi-isomorphism  $\mathrm{DR}_Y(\mathbf{R}f_* \mathrm{DR}_{X/Y}(\mathcal{M})) \cong \mathrm{DR}_Y(f_+ \mathcal{M}) \cong \mathbf{R}f_*(\mathrm{DR}_X(\mathcal{M}))$ , where  $\mathbf{R}f_*$  is the usual derived direct image for abelian sheaves.

**4.6 A final remark.** A naive approach to the comparison problem for direct images may be the following argument, which is very elementary. The existence of the morphisms of the Theorem 4.4 in the category of  $\mathcal{O}_Y$ -modules and the fact that they are isomorphisms follow from the Corollary 4.2.3. In order to conclude the proof, it is enough to identify the structure of  $\mathcal{D}_Y$ -module induced by the structure of the Gauss-Manin connection on the left with that of the right, and this may be done locally. The isomorphisms of Lemma 4.5.1 can be constructed locally in the following way. Using a local decomposition  $f^* \Omega_Y^1 \oplus \Omega_{X/Y}^1 \cong \Omega_X^1$ , we have (locally) that  $(f^* \Omega_Y^\bullet \otimes_{\mathcal{O}_X} \Omega_{X/Y}^\bullet)_{\mathrm{tot}} \cong \Omega_X^\bullet$  (see [B.A ch.III §7 n.7] as graded modules, and the usual signs convention identifies the differentials). In this isomorphism the Leray filtration on the right is identified with the filtration  $F_I$  of the total complex on the left. Therefore the Leray spectral sequence of  $\mathcal{M}$  for  $f$  is locally identified with the spectral sequence  $E_I$  of the bicomplex  $(f^* \Omega_Y^\bullet \otimes_{\mathcal{O}_X} \Omega_{X/Y}^\bullet(\mathcal{M}))$  with respect to the functor  $\mathbf{R}f_*$ . In particular we have at the  $E_1$  level that  $\mathrm{DR}_Y(R^q f_* \mathrm{DR}_{X/Y}(\mathcal{M})) \cong \mathrm{DR}_Y(\mathcal{H}^q(f_+ \mathcal{M}))$ , which identifies the Gauss-Manin connection with the connection induced on  $\mathcal{H}^q(f_+ \mathcal{M})$  by the  $\mathcal{D}_Y$ -module structure of  $f_+ \mathcal{M}$ , and concludes the “proof”. The unpleasant point in this argument, and probably the reason of its absence in the literature, is that it makes use of a non canonical local decomposition of the canonical exact sequence of differentials.

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